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Dynamical behavior of HIV immunology model with non-integer time fractional derivatives

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ABSTRACT

This paper introduces a mathematical model which describes the dynamics of the spread of HIV in the human body. Human immunodeficiency virus infection destroys the body immune system, increases the risk of certain pathologies, damages body organs such as the brain, kidney and heart or cause the death. Unfortunately, this infection disease currently has no cure to control the diseases. We propose a fractional order model in this paper to describe the dynamics of human immunodeficiency virus (HIV) infection. The Caputo fractional derivative operator of order $\alpha \in (0,1]$ is employed to obtain the system of fractional differential equations. The basic reproductive number is derived for a general viral production rate which determines the local stability of the infection free equilibrium. The solution of the time fractional model has been procured by employing Laplace Adomian decomposition method (LADM) and the accuracy of the scheme is presented by convergence analysis Moreover, numerical simulation are performed to study the dynamical behavior of solution of the models. Simulations of different epidemiological classes at the effect of the fractional parameters α revealed that most undergoing treatment join the recovered class. The results show the both viral production rate and death rate of infected cells play an important role the disease in the society.

1. Introduction

Human Immunodeficiency Virus (HIV) is responsible for AIDS which belongs to family of retroviruses. In United States during 1980s first case of immune deficiency syndrome occurred among homosexual men. Human immunodeficiency virus identified during 1983 by the etiological agent which is caused AIDS disease (CDC, 1999a-f). Another way of spreading HIV through sexual contact within the closed network among injection drug users, which are characterized by multiple sex partners, unprotected sexual intercourse and exchange of sex for money (Friedman et al., 1995). The inclusion of alcohol and other non-injection substances to this lethal mixture only increases the HIV/AIDS caseload (Grella et al., 1995; Word and Bowser, 1997). A major risk factor for HIV/AIDS among injection drug users is crack use; one study found that crack abusers reported more sexual partners in the last 12 months (Word and Bowser, 1997; Gao et al., 1999; Zhu et al., 1998).

The virus HIV is a reason for spread the AIDS and it is considered by severe decreases in CD4+T cells, which means a person who catch this disease grows a weak immune system and becomes susceptible to contracting life-mortal infections. AIDS occurs late in HIV disease. Many countries are strictly noticed HIV cases now and get the positive results against the infection in the early stages and because counting only AIDS cases is no longer satisfactory for projecting trends to pandemic (CDC, 1993; 1998). Since in recent years fractional calculus has attracted great attentions from researchers and different aspects of the said subject is under consideration for research. This is due to the fact that fractional derivative is important tool to explain the dynamical behavior of various physical systems. The strength of this differential operator is their nonlocal characteristics which do not exist in the integer order differential operators. The distinguished features of fractional differential equations are that it outlines memory and transmitted properties of numerous mathematical models. As a fact, that fractional order models are more realistic and practical than the classical integer order models.

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Fractional order derivative produces greater degree of freedom in these models. Arbitrary order derivatives are powerful tools for the discretion of the dynamical behavior of various biomaterial and systems. The most iterating feature of these models is their global (nonlocal) characteristics which do not exist in the classical order models (Arruda et al., 2015).

Laplace transform method is a useful technique in different field of biological science, engineering and applied mathematics. The coupling of ADM and Laplace transform leads to a powerful method known as Laplace Adomain decomposition method. With the help of Laplace transform, we convert a differential equation to an algebraic equation and the nonlinear terms are decomposed in terms of Adomain polynomials. The given numerical technique works powerfully for a system of deterministic as well as stochastic differential equations. More unambiguously, it can be used for classical as well as fractional order system of linear and nonlinear ordinary. Further it has no need of pre-defined step size like RK4. Also, this method does not depend upon on a parameter like needed for homotopy perturbation method (HPM) and homotopy analysis method (HAM). Although the solutions obtained vis this method are the same as obtained by ADM, for detail see (Haq et al., 2017; Jafari et al., 2011a;b; Zhu et al., 1998).

2. Mathematical model

In model formulation of HIV we uses different variables here x represents the compartment of susceptible cells (i.e., the compartment of those individuals which are not infected an able to catch the disease), y compartment represents the already infected cells, v represents the free viruses in the body, z represents the defense cells (CD8+T and B) and z_a corresponds to the activated defense cells (Jafari et al., 2011a). The system is given as

$$\frac{dx}{dt} = \lambda_x - \mu_x x - \beta_v xv \quad (1)$$
$$\frac{dy}{dt} = \beta_v xv - \mu_y y - P_y yZ_a \quad (2)$$
$$\frac{dv}{dt} = k_v \mu_y y - \mu_v v - P_v vZ_a \quad (3)$$
$$\frac{dz}{dt} = \lambda_z - \mu_z z - \beta_z zv \quad (4)$$
$$\frac{dz_a}{dt} = \beta_z zv - \mu_z z_a \quad (5)$$

with initial conditions:

$$x(0) = \frac{\lambda_x}{\mu_x}, y(0) = 0, v(0) = v_0, z(0) = \frac{\lambda_z}{\mu_z}, z_a(0) = 0 \quad (6)$$

Here uninfected cells x are formed at a constant supply rate λ_x and decay at rate μ_x. These cells are infected by the free viruses at rate β_v. As for the infected cells y, they are produced from the interaction of their uninfected counter parts and the viruses at rate β_v, decay at rate μ_y and are reduced by the activated defense cells at rate P_v. Free viruses v are made from infected cells at rate k_v, decay at rate μ_v and are eliminated by the means of the activated defense cells z_a at rate P_v. The defense cells, in turn are generated at a constant rate λ_z, and decay at rate μ_z and activated by the viruses at rate β_z. The activated defense cells are generated from the defense cells in the presence of the virus, at rate β_z, and decay at rate μ_z.

The purposed model of HIV in fractional differential equation (FDEs) is as follows

$$D^{\alpha_1}x(t) = \lambda_x - \mu_x x - \beta_v xv \quad (7)$$
$$D^{\alpha_2}y(t) = \beta_v xv - \mu_y y - P_y yZ_a \quad (8)$$
$$D^{\alpha_3}v(t) = k_v \mu_y y - \mu_v v - P_v vZ_a \quad (9)$$
$$D^{\alpha_4}z(t) = \lambda_z - \mu_z z - \beta_z zv \quad (10)$$
$$D^{\alpha_5}z_a(t) = \beta_z zv - \mu_z z_a \quad (11)$$

with following initial conditions

$$let \quad x(0) = N_1, y(0) = N_2, v(0) = N_3, z(0) = N_4, z_a(0) = N_5 \quad (12)$$

For this model the initial conditions are not independent, since they must satisfy the condition $N_1 + N_2 + N_3 + N_4 + N_5 = N$ where N is the total population in the system.

3. Preliminaries

In this section, we give some fundamental results and definitions from fractional calculus. For detailed over view of the topic readers are referred to (Haq et al., 2017; Johnston et al., 2016; Jafari et al., 2011a; 2011b).

Definition 3.1: The Riemann-Liouville fractional integration of order α is defined as:

$$\left(J_{t_0}^{\alpha} f\right)(t) = \frac{1}{\alpha}\int_{t_0}^{\alpha}(t-s)^{\alpha-1} f(s)ds, \qquad \alpha > 0, t > t_0$$
$$\left(J_{t_0}^{\alpha} f\right)(t) = f(t)$$

The Riemann-Liouville derivative has certain disadvantages such that the fractional derivative of a constant is not zero. Therefore, we will make use of Caputo's definition owing to its convenience for initial conditions of the fractional differential equations.

Definition 3.2: The Riemann-Liouville fractional integration of order α is defined as:

$$D^{\alpha}f(t) = D^n(J^{n-\alpha}f(t)),$$
$$D_*^{\alpha}f(t) = (J^{n-\alpha}(D^n f(t)),$$

where $n-1 < \alpha \leq \alpha \leq n, n \in N$, f is the given function, It is known that $(J_{t_0}^{\alpha} f)(t) \to f(t)$ as $\alpha \to 1$.

Definition 3.3: The definitions of Laplace transform of Caputo's derivative and Mittag-Leffler function in two arguments is written as

$$L\{D^{\alpha}f(t)\} = s^{\alpha}F(s) - \sum_{i=0}^{n-1} s^{\alpha-i-1} f^i(0), \qquad n-1 < \alpha \leq n, n \in N$$

4. Mathematical analysis

Disease Free Equilibrium: To evaluate the equilibrium point, we take

$$D^{\alpha_1}x(t) = D^{\alpha_2}y(t) = D^{\alpha_3}v(t) = D^{\alpha_4}z(t) = D^{\alpha_5}(z)_a(t) = 0 \quad (13)$$

When naturally, the disease die out then the solution of the above system asymptotically approaches a disease free population or equilibrium is of the form i.e., $E_0 = (x, y, v, z, z_a) = (\frac{\lambda_x}{\mu_x}, 0,0, \frac{\lambda_z}{\mu_z}, 0)$.

Theorem 4.1: The disease-free equilibrium$E_0$is locally asymptotically stable if $R_0 < 1$ and is unstable otherwise (Arruda et al., 2015).

Reproductive number: In this system the threshold result of this equilibrium is $R_0 = \frac{\lambda_x \beta_v k_v}{\mu_x \mu_v} < 1$, so this is in disease free state.

Non negative solution: Let $R_+^5 = \{w \in R^5, w \geq 0\}$ and $w(t) = \big(x(t), y(t), v(t), z(t), z_a(t)\big)^T$.

Lemma 1: Let $h(x) \in C \in [a, b]$ and $D^\alpha h(x) \in C[a, b]$ for $0 < \alpha \leq 1$ then, we have $h(s) = h(a) + \frac{1}{(\alpha+1)!} D^\alpha h(\eta)(x - \alpha)^\alpha$ with $0 \leq \eta \leq w$ for all $w \in (a, b]$.

Theorem 4.2: There is a unique solution for the initial value problem given by (7)-(11), and the solution remains in $R^5, w \geq 0$.

Proof: The uniqueness and existence for the solution of (7)-(11), in $(0, \alpha)$. Our aim is to show the domain $R^5, w \geq 0$ is positively invariant. Since

$$D^{\alpha_1} x|_{x=0} = \lambda_x \geq 0$$
$$D^{\alpha_2} y|_{y=0} = \beta_v xv \geq 0$$
$$D^{\alpha_3} v|_{v=0} = k_v \mu_y y \geq 0$$
$$D^{\alpha_4} z|_{z=0} = \lambda_z \geq 0$$
$$D^{\alpha_5} (z)_a|_{(z)_a=0} = \beta_z zv \geq 0$$

The nonnegative solution satisfied the vector field points intoR_+^5.

5. The Laplace-Adomian decomposition method

Consider the fractional-order epidemic model (7) - (11) subject to the initial condition (12). The nonlinear terms in this model is xv, yz_a, vz_a, zv and $\lambda_x, \mu_x, \beta_v, \mu_y, P_y k_v, \mu_v, P_v, \mu_z, \lambda_z, \beta_z$are known constants. For$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 1$ the fractional order model converts to the classical epidemic model. By using Laplace transform on system (7) - (11), we get

$$L\{D^{\alpha_1} x(t)\} = \lambda_x L\{1\} - \mu_x L\{x\} - \beta_v L\{xv\} \quad (14)$$
$$L\{D^{\alpha_2} y(t)\} = \beta_v L\{xv\} - \mu_y L\{y\} - P_y L\{yz_a\} \quad (15)$$
$$L\{D^{\alpha_3} v(t)\} = k_v L\{\mu_y y\} - \mu_v L\{v\} - P_v L\{vz_a\} \quad (16)$$
$$L\{D^{\alpha_4} z(t)\} = \lambda_z L\{1\} - \mu_z L\{z\} - \beta_z L\{zv\} \quad (17)$$
$$L\{D^{\alpha_5} z_a(t)\} = \beta_z L\{zv\} - \mu_z L\{z_a\}. \quad (18)$$

Using property of Laplace transform, we get

$$S^{\alpha_1} L\{x\} - S^{\alpha_1 - 1} x(0) = \frac{\lambda_x}{S} - \mu_x L\{x\} - \beta_v L\{xv\} \quad (19)$$
$$S^{\alpha_2} L\{y\} - S^{\alpha_2 - 1} y(0) = \beta_v L\{xv\} - \mu_y L\{y\} - P_y L\{yZ_a\} \quad (20)$$
$$S^{\alpha_3} L\{v\} - S^{\alpha_3 - 1} v(0) = k_v L\{\mu_y y\} - \mu_v L\{v\} - P_v L\{vZ_a\} \quad (21)$$
$$S^{\alpha_4} L\{z\} - S^{\alpha_4 - 1} z(0) = \frac{\lambda_z}{S} - \mu_z L\{z\} - \beta_z L\{zv\} \quad (22)$$
$$S^{\alpha_5} L\{z_a\} - S^{\alpha_5 - 1} z_a(0) = \beta_z L\{zv\} - \mu_z L\{z_a\}. \quad (23)$$

Using initial conditions we have:

$$L\{x\} = \frac{x(0)}{s} + \frac{\lambda_x}{s^{\alpha_1 - 1}} - \frac{\mu_x}{s^{\alpha_1}} L\{x\} - \frac{\beta_v}{s^{\alpha_1}} L\{xv\} \quad (24)$$
$$L\{y\} = \frac{y(0)}{s} + \frac{\beta_v}{s^{\alpha_2}} L\{xv\} - \frac{\mu_y}{s^{\alpha_2}} L\{y\} - \frac{p_y}{s^{\alpha_2}} L\{yZ_a\} \quad (25)$$
$$L\{v\} = \frac{v(0)}{s} + \frac{k_v}{s^{\alpha_3}} L\{\mu_y y\} - \frac{\mu_v}{s^{\alpha_3}} L\{v\} - \frac{p_v}{s^{\alpha_3}} L\{vZ_a\} \quad (26)$$
$$L\{z\} = \frac{z(0)}{s} + \frac{\lambda_z}{s} - \frac{\mu_z}{s^{\alpha_4}} L\{z\} - \frac{\beta_z}{s^{\alpha_4}} L\{zv\} \quad (27)$$
$$L\{z_a\} = \frac{z_a(0)}{S} + \frac{\beta_z}{s^{\alpha_5}} L\{zv\} - \frac{\mu_z}{s^{\alpha_5}} L\{z_a\}. \quad (28)$$

The method assumes the solution as an infinite series:

$$x = \sum_{k=0}^{\infty} x_k \ , y = \sum_{k=0}^{\infty} y_k \ , v = \sum_{k=0}^{\infty} v_k \ , z = \sum_{k=0}^{\infty} z_k \ , z_a = \sum_{k=0}^{\infty} (z_a)_k$$

The non-linearity xv, yz_a, vz_a, zv are decomposed as:

$$xv = \sum_{k=0}^{\infty} A_k \ , yz_a = \sum_{k=0}^{\infty} B_k \ , vz_a = \sum_{k=0}^{\infty} C_k \ , zv = \sum_{k=0}^{\infty} D_k$$

where A_k, B_k, C_k, D_k are so called Adomian polynomials given as:

$$A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} \Big| \Big[\sum_{j=0}^{k} \lambda^j x_j \sum_{j=0}^{k} \lambda^j v_j\Big]\Big|_{\lambda=0}$$
$$B_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} \Big| \Big[\sum_{j=0}^{k} \lambda^j y_j \sum_{j=0}^{k} \lambda^j (z_a)_j\Big]\Big|_{\lambda=0}$$
$$C_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} \Big| \Big[\sum_{j=0}^{k} \lambda^j v_j \sum_{j=0}^{k} \lambda^j (z_a)_j\Big]\Big|_{\lambda=0}$$
$$D_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} \Big| \Big[\sum_{j=0}^{k} \lambda^j z_j \sum_{j=0}^{k} \lambda^j v_j\Big]\Big|_{\lambda=0}$$

now from equation the required result is:

$$L\{x_0\} = \frac{N_1}{S} + \frac{\lambda_x}{S^{\alpha_1 - 1}} \ , L\{x_1\} = -\frac{\mu_x}{S^{\alpha_1}} L\{x_0\} - \frac{\beta_v}{s^{\alpha_1}} L\{A_0\}, \dots, L\{x_{k+1}\} = -\frac{\mu_x}{S^{\alpha_1}} L\{x_k\} - \frac{\beta_v}{s^{\alpha_1}} L\{A_k\} \quad (29)$$

$$L\{y_0\} = \frac{N_2}{S}, L\{y_1\} = \frac{\beta_v}{s^{\alpha_2}} L\{A_0\} - \frac{\mu_y}{s^{\alpha_2}} L\{y_0\} - \frac{p_y}{s^{\alpha_2}} L\{B_0\}, \dots, L\{y_{k+!}\} = \frac{\beta_v}{s^{\alpha_2}} L\{A_k\} - \frac{\mu_y}{s^{\alpha_2}} L\{y_k\} - \frac{p_y}{s^{\alpha_2}} L\{B_k\} \quad (30)$$

$$L\{v_0\} = \frac{N_3}{S}, L\{v_1\} = \frac{k_v \mu_y}{s^{\alpha_3}} L\{y_0\} - \frac{\mu_v}{s^{\alpha_3}} L\{v_0\} - \frac{p_v}{s^{\alpha_3}} L\{C_0\}, \dots. \quad L\{v_{k+1}\} = \frac{k_v \mu_y}{s^{\alpha_3}} L\{y_k\} - \frac{\mu_v}{s^{\alpha_3}} L\{v_k\} - \frac{p_v}{s^{\alpha_3}} L\{C_k\} \quad (31)$$

$$L\{z_0\} = \frac{N_4}{S} + \frac{\lambda_z}{S^{\alpha_4 - 1}}, \ L\{z_1\} = -\frac{\mu_z}{s^{\alpha_4}} L\{z_0\} - \frac{\beta_z}{s^{\alpha_4}} L\{D_0\}, \dots, L\{z_{k+1}\} = -\frac{\mu_z}{s^{\alpha_4}} L\{z_k\} - \frac{\beta_z}{s^{\alpha_4}} L\{D_k\} \quad (32)$$

$$L\{(Z_a)_0\} = \frac{N_5}{S}, L\{(Z_a)_1\} = \frac{\beta_z}{s^{\alpha_5}} L\{D_1\} - \frac{\mu_z}{s^{\alpha_5}} L\{(Z_a)_0\}, \dots, L\{(Z_a)_{k+1}\} = \frac{\beta_z}{s^{\alpha_5}} L\{D_k\} - \frac{\mu_z}{s^{\alpha_5}} L\{(Z_a)_k\} \quad (33)$$

The aim is to study the mathematical behavior of the solution $x(t), y(t), v(t), z(t), z_a(t)$ for the different values of α. by applying the inverse Laplace transform to both sides of above equations. We computed the first three terms:

$$x_0 = 1000 + 20\frac{t^{\alpha_1}}{\alpha_{1!}} \ , y_0 = 0 \ , v_0 = 10^{-3} \ , z_0 = 500 + 20.20\frac{t^{\alpha_4}}{\alpha_{4!}} \ , (Z_a)_0 = 0$$

$$x_1 = -\frac{20.000024t^{\alpha_1}}{\Gamma(\alpha_1+1)} - \frac{0.40000048t^{2\alpha_1}}{\Gamma(2\alpha_1+1)}, \qquad y_1 = \frac{2.4\times10^{-5}t^{\alpha_2}}{\Gamma(\alpha_2+1)} - \frac{4.8\times10^{-7}t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}, \quad v_1 = -\frac{2.4\times10^{-3}t^{\alpha_3}}{\Gamma(\alpha_3+1)}$$

$$z_1 = -\frac{20.0000025t^{\alpha_4}}{\Gamma(\alpha_4+1)} - \frac{0.800000t^{2\alpha_4}}{\Gamma(2\alpha_4+1)}, \qquad (z_a)_1 = \frac{2.5\times10^{-6}t^{\alpha_5}}{\Gamma(\alpha_5+1)} - \frac{0.40000048t^{\alpha_4+\alpha_5}}{\Gamma(\alpha_4+\alpha_5+1)}$$

$$x_2 = \frac{0.400000528t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} + \frac{9.60801153t^{3\alpha_1}}{\Gamma(3\alpha_1+1)} + \frac{5.76\times10^{-5}t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1+\alpha_3+1)} + \frac{1.152\times10^{-6}(\alpha_1+\alpha_3)!t^{2\alpha_1+\alpha_3}}{\alpha_1!\alpha_3!\Gamma(2\alpha_1+\alpha_3+1)}$$

$$y_2 = -\frac{5.76\times10^{-5}t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} - \frac{5.76\times10^{-6}t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} + \frac{1.152\times10^{-7}t^{\alpha_1+2\alpha_2}}{\Gamma(\alpha_1+2\alpha_2+1)} - \frac{1.152\times10^{-6}(\alpha_1+\alpha_3)!t^{\alpha_1+\alpha_2+\alpha_3}}{\alpha_1!\alpha_3!\Gamma(\alpha_1+\alpha_2+\alpha_3+1)\Gamma(\alpha_3+1)} - \frac{4.48\times10^{-7}t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} - \frac{9.60001152\times10^{-9}t^{2\alpha_1+\alpha_2}}{\Gamma(2\alpha_1+\alpha_2+1)}$$

$$v_2 = \frac{2.0736\times10^{-3}t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} + \frac{5.76\times10^{-3}t^{2\alpha_3}}{\Gamma(2\alpha_3+1)} - \frac{5\times10^{-11}t^{\alpha_3+\alpha_5}}{\Gamma(\alpha_3+\alpha_5+1)} + \frac{4.1472\times10^{-5}t^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} - \frac{2\times10^{-12}t^{\alpha_3+\alpha_4+\alpha_5}}{\Gamma(\alpha_3+\alpha_4+\alpha_5+1)}$$

$$z_2 = \frac{0.8000011t^{2\alpha_4}}{\Gamma(2\alpha_4+1)} + \frac{0.032000008\ t^{3\alpha_4}}{\Gamma(3\alpha_4+1)} + \frac{2.4\times10^{-7}(\alpha_3+\alpha_4)!t^{\alpha_3+2\alpha_4}}{\alpha_4!\alpha_3!\Gamma(\alpha_3+2\alpha_4+1)} + \frac{6\times10^{-6}t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}$$

$$(z_a)_2 = \frac{6\times10^{-6}t^{\alpha_3+\alpha_5}}{\Gamma(\alpha_3+\alpha_5+1)} - \frac{4.0000005\times10^{-9}\ t^{2\alpha_4+\alpha_5}}{\Gamma(2\alpha_4+\alpha_5+1)} - \frac{2.4\times10^{-7}(\alpha_3+\alpha_4)!t^{\alpha_3+\alpha_4+\alpha_5}}{\alpha_4!\alpha_3!\Gamma(\alpha_3++\alpha_4+\alpha_5+1)} - \frac{1\times10^{-7}t^{\alpha_5+\alpha_4}}{\Gamma(\alpha_5+\alpha_4+1)} - \frac{1\times10^{-7}t^{2\alpha_5}}{\Gamma(2\alpha_5+1)} - \frac{4\times10^{-9}\ t^{\alpha_4+2\alpha_5}}{\Gamma(\alpha_4+2\alpha_5+1)}$$

The series solution for fractional order differential equation is as follows:

$$x(t) = 1000 - \frac{0.000024t^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{4.8\times10^{-8}t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} + \frac{9.60801153t^{3\alpha_1}}{\Gamma(3\alpha_1+1)} + \frac{5.76\times10^{-5}t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1+\alpha_3+1)} + \frac{1.152\times10^{-6}(\alpha_1+\alpha_3)!t^{2\alpha_1+\alpha_3}}{\alpha_1!\alpha_3!\Gamma(2\alpha_1+\alpha_3+1)} + \cdots \quad (34)$$

$$y(t) = \frac{2.4\times10^{-5}t^{\alpha_2}}{\Gamma(\alpha_2+1)} - \frac{9.28\times10^{-7}t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} - \frac{5.76\times10^{-5}t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} - \frac{5.76\times10^{-6}t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} + \frac{1.152\times10^{-7}t^{\alpha_1+2\alpha_2}}{\Gamma(\alpha_1+2\alpha_2+1)} - \frac{1.152\times10^{-6}(\alpha_1+\alpha_3)!t^{\alpha_1+\alpha_2+\alpha_3}}{\alpha_1!\alpha_3!\Gamma(\alpha_1+\alpha_2+\alpha_3+1)\Gamma(\alpha_3+1)} - \frac{9.60001152\times10^{-9}t^{2\alpha_1+\alpha_2}}{\Gamma(2\alpha_1+\alpha_2+1)} + \cdots \quad (35)$$

$$v(t) = 10^{-3} - \frac{2.4\times10^{-3}t^{\alpha_3}}{\Gamma(\alpha_3+1)} + \frac{2.0736\times10^{-3}t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} + \frac{5.76\times10^{-3}t^{2\alpha_3}}{\Gamma(2\alpha_3+1)} - \frac{5\times10^{-11}t^{\alpha_3+\alpha_5}}{\Gamma(\alpha_3+\alpha_5+1)} + \frac{4.1472\times10^{-5}t^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)} - \frac{2\times10^{-12}t^{\alpha_3+\alpha_4+\alpha_5}}{\Gamma(\alpha_3+\alpha_4+\alpha_5+1)} + \cdots \quad (36)$$

$$z(t) = 500 + 20.20\frac{t^{\alpha_4}}{\alpha_4!} - \frac{20.0000025t^{\alpha_4}}{\Gamma(\alpha_4+1)} - \frac{1.1\times10^{-6}t^{2\alpha_4}}{\Gamma(2\alpha_4+1)} + \frac{0.032000008\ t^{3\alpha_4}}{\Gamma(3\alpha_4+1)} + \frac{2.4\times10^{-7}(\alpha_3+\alpha_4)!t^{\alpha_3+2\alpha_4}}{\alpha_4!\alpha_3!\Gamma(\alpha_3+2\alpha_4+1)} + \frac{6\times10^{-6}t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)} + \cdots \quad (37)$$

$$(z_a)(t) = \frac{2.5\times10^{-6}t^{\alpha_5}}{\Gamma(\alpha_5+1)} - \frac{0.40000048t^{\alpha_4+\alpha_5}}{\Gamma(\alpha_4+\alpha_5+1)} + \frac{0.8000011t^{2\alpha_4}}{\Gamma(2\alpha_4+1)} + \frac{0.032000008\ t^{3\alpha_4}}{\Gamma(3\alpha_4+1)} + \frac{2.4\times10^{-7}(\alpha_3+\alpha_4)!t^{\alpha_3+2\alpha_4}}{\alpha_4!\alpha_3!\Gamma(\alpha_3+2\alpha_4+1)} + \frac{6\times10^{-6}t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)} + \cdots \quad (38)$$

Table 1 is parameter which used in the model and given in (Arruda et al., 2015)

6. Numerical results and discussion

The numerical results of susceptible cells, infected cells, free virus in a body, defense cells and activated defense cells are established by using LADM. To observe the effect of parameters suing in model and given in Table 1 with initial conditions $(0) = 1000mm,\ y(0) = 0,\ v(0) = 10^{-3}mm,\ z(0) = 500\ mm,\ Z_a(0) = 0$. For the reliable investigation, evaluation is made for different values of α. From Figs. 1–5, we observe that fractional order HIV Immunology model has more degree of freedom as compared to ordinary derivatives.

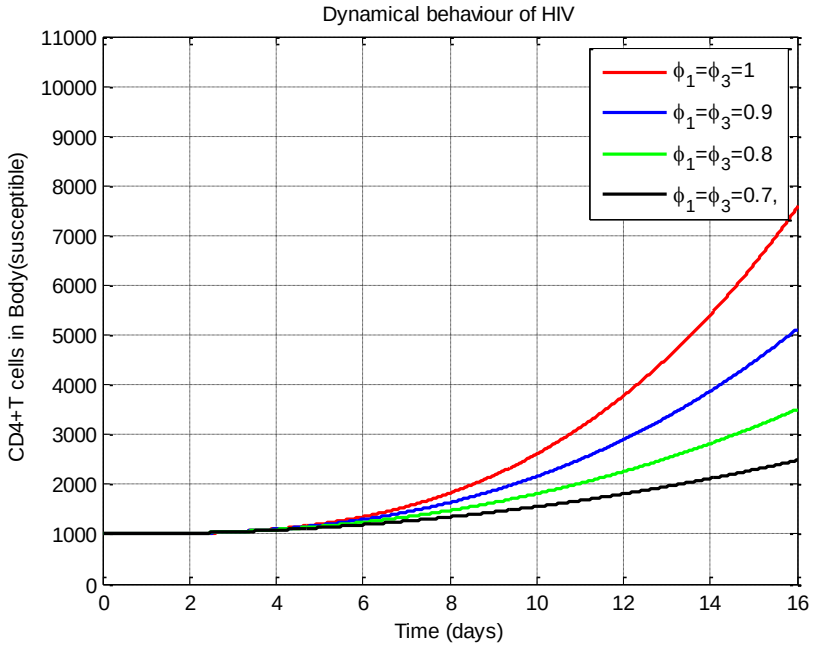

Fig.1: Numerical solution for susceptible cells $x(t)$ in a time t (days) at different values of α

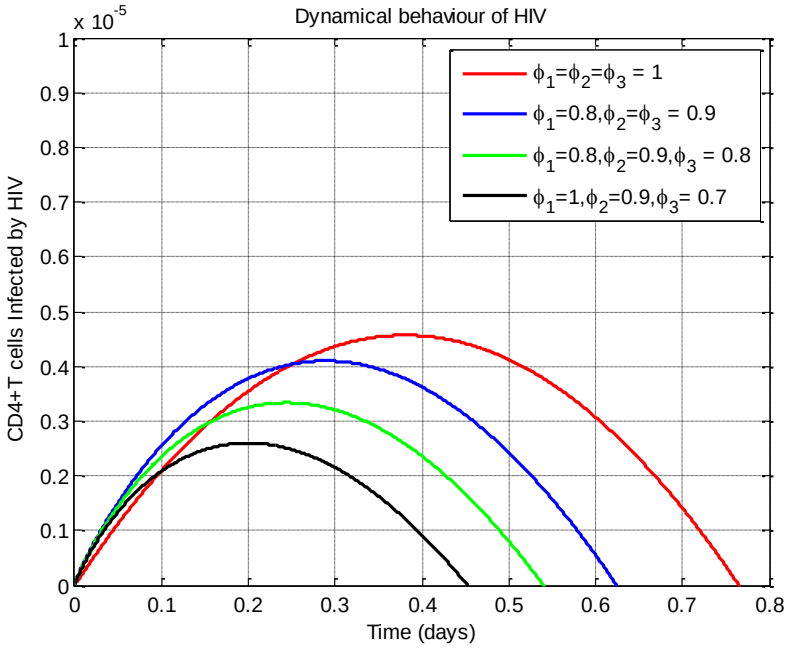

Fig. 2: Numerical solution for Infected cells $y(t)$ in a time t (days) at different values of α

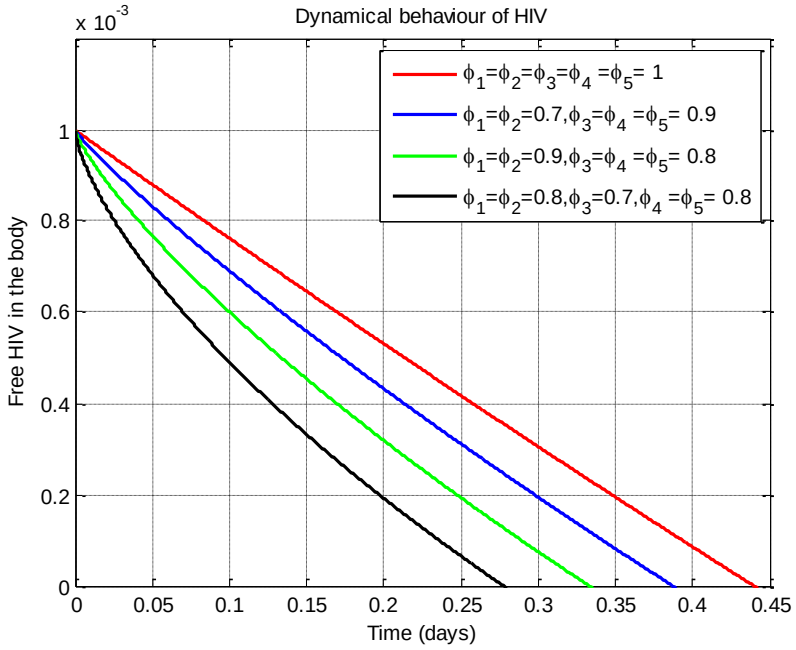

Fig. 3: Numerical solution for free virus in a body v(t) in a time t (days) at different values of α

By taking non-integer values of fractional parameter, remarkable responses of the compartments of the proposed model are obtained. Another remarkable point to be considered that we used small interval of time because we have assumed comparatively small initial values. For large interval of time, the initial values to data are taken large so that the population may not be negative. For different values of α solution converges to steady state and gives the better convergence by decreasing the fractional values of α.

7. Convergence analysis

The obtained series solution is rapidly convergent and also converges uniformly to the exact solution. We use the classical techniques to verify the convergence of the series (34-38). We check the condition of convergence of the method by using the idea of the following theorem (Abdelrazec and Pelinovsky, 2011; Naghipour and Manafian, 2015).

Theorem.7.1: Let W be a Banach space and $F: W \to W$ be a contractive nonlinear operator then there exit $w, w' \in W, \|F(w) - F(w')\| \leq k\|w - w'\|, 0 < k < 1$. Then F has a unique point w such that, $Fw = w$ where $w = \big(x(t), y(t), v(t), z(t), z_a(t)\big)$. The series given in (34-38) by using ADM technique is given as:

$w_n = Tw_{n-1}, w_{n-1} = \sum_{j=1}^{n-1} w_j \quad , n = 1,2,3, \ldots$

and suppose that $w_0 \in B_r(w)$ where $B_r(w) = \{w' \in W: \|w' - w\| < r\}$ then we get

(i) $w_n \in B_r(w)$
(ii) $\lim_{n\to\infty} w_n = w$

Proof: For (i) by using mathematical induction for n=1, we obtained

$$\|w_0 - w\| = \|F(w_0) - F(x)\| \leq k\|w_0 - w\|$$

suppose that the statement is true for $m-1$ then,

$$\|w_0 - w\| \leq k^{m-1}\|w_0 - w\|$$

we get

$$\|w_m - w\| = \|F(w_{m-1}) - F(x)\| \leq k\|w_{m-1} - w\| \leq k^m\|w_0 - w\|$$
$$\|w_m - w\| \leq k^n\|w_0 - w\| \leq k^n r \leq r$$
which implies that $w_n \in B_r(w)$

(ii) Since

$$\|w_m - w\| \leq k^n\|w_0 - w\| \; and \; \lim_{n\to\infty} k^n = 0$$

therefore, we have the $\lim_{n\to\infty}\|w_n - w\| = 0 \Rightarrow \lim_{n\to\infty} w_n = w$

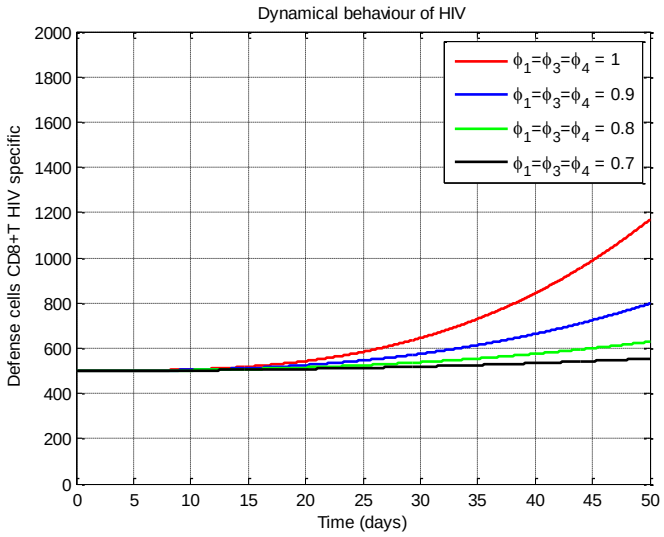

Fig. 4: Numerical solution for defense cells $z(t)$ in a time t (days) at different values of α

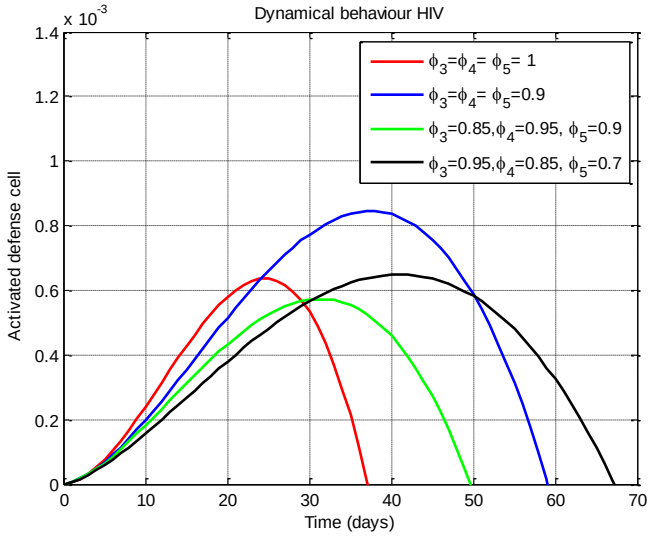

Fig. 5: Numerical solution for activated defense cells $z_a(t)$ in a time t (days) at different values of α

8. Conclusion

In this paper, we developed a scheme for numerical solution of epidemic fractional HIV Immunology model by using Laplace Adomian decomposition method. The well-known epidemic model namely susceptible cells, infected cells ,free virus in a body, defense cells and activated defense cells is considered with and without demographic effects. The model represents population dynamics during the disease as a set of non-linear coupled ordinary differential equations. There is no exact solution available in the literature for this model up to the best of author's knowledge. It is observed that

the infection rate and reproductive numbers play a key role for an epidemic to occur and the epidemic can be controlled by vaccination. It is also observed that to eliminate the disease, it is not necessary to vaccinate whole of the population. The efficiency and accuracy of the proposed scheme is provided by performing convergence analysis. The effect of fractional parameter on our obtained solutions is presented through Tables and graphs. It is worthy to observe that fractional derivatives show significant changes and memory effects as compared to ordinary derivatives.

Table 1: Parameters values of HIV model

Parameters	Values	Parameters	Values
μ_x	0.02	β_z	5×10^{-6}
μ_y	0.24	β_v	2.4×10^{-5}
μ_v	2.4	p_y	0.02
μ_z	0.04	p_v	0.02
k_v	360	λ_x	20
λ_z	20		

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